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IDENTITIES, CONTINUED FRACTIONS AND PARTITION THEORETIC INTERPRETATIONS

Sunil Singh, Mohmmad Shahjade* and Vijay Yadav**

Department of Mathematics,
Dr. Homi Bhabha State University,
The Institute of Science, Mumbai - 32, INDIA

E-mail : sunilsingh@iscm.ac.in

*Department of Mathematics,
MANUU Poly. 8th Cross, 1st Stage,
3rd Block, Nagarbhavi, Banglore - 72 INDIA

E-mail : mohammadshahjade@gmail.com

**Department of Mathematics,
SPDT College,
Andheri (East), Mumbai - 59, INDIA

E-mail : vijaychottu@yahoo.com

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Abstract: In this paper, taking two identities of Rogers-Ramanujan type, a continued fraction has been established and partition theoretic interpretations of the identities have also been given with proofs.

Keywords and Phrases: Identity, continued fraction, partition theoretic interpretation.

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1. Introduction, Notations and Definitions

Recall the usual q-series notations,

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad n > 0,$$

$$(a;q)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \text{and} \quad (a;q)_0 = 1,$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

$$(a_1, a_2, ..., a_k; q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n$$

and

$$(a_1, a_2, ..., a_k; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty...(a_k;q)_\infty.$$

The study of finite continued fractions i.e., expressions of the form

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}}$$

which is written more economically as,

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n}$$

began in its explicit form in the letter decades of the 16th century with a paper of Bombelli written when the concepts and notations of algebra were first being laid down in Italy and France.

Thus use of continued fractions as an important tool in number theory began with the 17th century results of Schwenter, Huygens and Walls and come to maturity with the work of Euler in 1737. The infinite continued fractions is written as,

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n+}\cdots\infty$$

One of the old result on continued fraction is,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots\infty$$

Proof of this result is very simple, viz.,

$$\frac{\sqrt{5}-1}{2} = \cfrac{1}{\cfrac{2}{\sqrt{5}-1}\left(\cfrac{\sqrt{5}+1}{\sqrt{5}+1}\right)}$$

$$= \frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{1}{1+\frac{\sqrt{5}-1}{2}}$$

Now, iterating this process one can get above result on continued fraction.
This result might has attracted Ramanujan. He has established large number of results involving continued fraction in his first, second, third and also in 'lost' Notebooks. Many other mathematicians all over the world have either established or proved the results of Srinivasa Ramanujan. The well known and highly celebrated identities due to Rogers-Ramanujan are

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.2}$$

[See 4; p. 86 and 87]

Rogers-Ramanujan also proved that

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}$$

$$= \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\ldots_\infty, \quad \text{where } |q|<1. \tag{1.3}$$

[See 4; corollary (6.2.6) p. 153]

Some important works on continued fractions are being quoted here Agarwal [1], Andrews and Bowmen [3], Benrdt, B. C. [5], Bhargava and Adiga [6], Denis R. Y. [7], Adiga, C., Berndt, B. C., Bhargava, S. and Watson, G. N. [2], Hirschhorn, M. D. [8], Masson, D. R. [9], Ramanathan, K. G. [10] and Singh, S. N. [11], Singh Sunil, Singh Satya Prakash and Yadav Vijay [12], Yadav Vijay and Dubey D. M. [15].
The aim of the present paper is to discuss about the following identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \frac{(-q^2;q^2)_\infty(q,q^4,q^5;q^5)_\infty}{(q^2;q^2)_\infty} \tag{1.4}$$

[13; (17)]

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_n}=\frac{(-q^2;q^2)_\infty(q^2,q^3,q^5;q^5)_\infty}{(q^2;q^2)_\infty}. \qquad (1.5)$$

[13; (99)]

Taking the ratio of (1.4) and (1.5) we find

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_n}}=\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}$$

$$=\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\;\ldots\infty, \quad \text{where } |q|<1. \qquad (1.6)$$

2. Identities and continued fraction

In this section we have established a continued fraction different from (1.6) for the ratio of (1.4) and (1.5).
Taking the ratio of the left hand sides of (1.4) and (1.5) we have

$$=\frac{\displaystyle\frac{1}{(1+q)}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q^3;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_n}} \qquad (2.1)$$

$$=\frac{\dfrac{1}{(1+q)}}{1+\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n}\left\{\frac{1}{(-q;q^2)_n}-\frac{1}{(-q^3;q^2)_n}\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q^3;q^2)_n}}}. \qquad (2.2)$$

On simplification, (2.2) gives,

$$= \frac{\dfrac{1}{(1+q)}}{1 - \dfrac{\displaystyle\sum_{n=1}^{\infty} \frac{q^{n^2+n+1}}{(q^2;q^2)_{n-1}(-q;q^2)_{n+1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q^3;q^2)_n}}},$$

$$= \frac{\dfrac{1}{(1+q)}}{1 - \dfrac{\dfrac{q^3}{(1+q)(1+q^3)}}{\dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q^3;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^5;q^2)_n}}}}. \tag{2.3}$$

Again,

$$= \frac{\dfrac{1}{(1+q)}}{1 - \dfrac{\dfrac{q^3}{(1+q)(1+q^3)}}{1 + \dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n}\left\{\frac{1}{(-q^3;q^2)_n} - \frac{q^{2n}}{(-q^5;q^2)_n}\right\}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^5;q^2)_n}}}},$$

$$= \cfrac{\cfrac{1}{(1+q)}}{1-\cfrac{\cfrac{q^3}{(1+q)(1+q^3)}}{1+\cfrac{\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_{n-1}(-q^3;q^2)_{n+1}}}{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^5;q^2)_n}}}},$$

which on simplification gives,

$$= \cfrac{\cfrac{1}{(1+q)}}{1-\cfrac{\cfrac{q^3}{(1+q)(1+q^3)}}{1+\cfrac{\cfrac{q^2}{(1+q^3)(1+q^5)}}{\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^5;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^7;q^2)_n}}}}}. \tag{2.4}$$

Proceeding similarly we have,

$$= \cfrac{\cfrac{1}{(1+q)}}{1-\cfrac{\cfrac{q^3}{(1+q)(1+q^3)}}{1+\cfrac{\cfrac{q^2}{(1+q^3)(1+q^5)}}{1-\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}}{(q^2;q^2)_n}\left\{\frac{1}{(-q^7;q^2)_n}-\frac{1}{(-q^5;q^2)_n}\right\}}{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^7;q^2)_n}}}}},$$

$$= \cfrac{\frac{1}{(1+q)}}{1 - \cfrac{\frac{q^3}{(1+q)(1+q^3)}}{1 + \cfrac{\frac{q^2}{(1+q^3)(1+q^5)}}{1 - \cfrac{\sum_{n=1}^{\infty} \frac{q^{n^2+3n+5}}{(q^2;q^2)_{n-1}(-q^5;q^2)_{n+1}}}{\sum_{n=0}^{\infty} \frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^7;q^2)_n}}}}},$$

which gives

$$= \cfrac{\frac{1}{(1+q)}}{1 - \cfrac{\frac{q^3}{(1+q)(1+q^3)}}{1 + \cfrac{\frac{q^2}{(1+q^3)(1+q^5)}}{1 - \cfrac{\frac{q^9}{(1+q^5)(1+q^7)}}{\cfrac{\sum_{n=0}^{\infty} \frac{q^{n^2+3n}}{(q^2;q^2)_n(-q^7;q^2)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2+5n}}{(q^2;q^2)_n(-q^9;q^2)_n}}}}}} \tag{2.5}$$

Iterating the process, we finally get,

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}}}{\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_n}} = \frac{1/(1+q)}{1-}\,\frac{q^3/(1+q)(1+q^3)}{1+}\,\frac{q^2/(1+q^3)(1+q^5)}{1-}$$
$$\frac{q^9/(1+q^5)(1+q^7)}{1+}\,\frac{q^4/(1+q^7)(1+q^9)}{1-}\,\frac{q^{15}/(1+q^9)(1+q^{11})}{1+}\,\ldots. \tag{2.6}$$

Now, applying [14; (2.3.14) p. 33] we get

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_n}} = \frac{1}{(1+q)-}\,\frac{q^3}{(1+q^3)+}\,\frac{q^2}{(1+q^5)-}\,\frac{q^9}{(1+q^7)+}\,\frac{q^4}{(1+q^9)-}\,\frac{q^{15}}{(1+q^{11})+}\,\ldots. \tag{2.7}$$

Comparing (1.6) and (2.7) we have

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\ldots = \frac{1}{(1+q)-}\,\frac{q^3}{(1+q^3)+}\,\frac{q^2}{(1+q^5)-}\,\frac{q^9}{(1+q^7)+}\,\frac{q^4}{(1+q^9)-}\,\frac{q^{15}}{(1+q^{11})+}\,\ldots. \tag{2.8}$$

3. Partition theoretic interpretation

In this section partition theoretic interpretations of identities (1.4) and (1.5) have been discussed.

Identities (1.4) and (1.5) can be written after some simple manipulation as,

$$\sum_{n=0}^{\infty}\frac{(-q^{2n+3};q^2)_\infty q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{3.1}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q^{2n+1};q^2)_\infty q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{3.2}$$

Interpretation of (3.1)

The number of partitions of a positive integer k into n even parts and distinct odd parts in which no one is less than $(2n+3)$, equals the number of partitions of k into parts $\equiv 2$ or 3 (modulo 5).

Proof. Let the partition of positive integer k be written as

$$k = a_1 + a_2 + a_3 + \ldots + a_n + b_1 + b_2 + b_3 + \ldots, \tag{3.3}$$

where parts are in ascending order and $a_1, a_2, \ldots, a_n$ are even numbers where as $b_1, b_2, b_3, \ldots$ are distinct odd parts in which $b_1 \geq 2n+3$.

Thus we have

$$\begin{aligned} a_1 &\geq 2 \\ a_2 &\geq 4 \\ a_3 &\geq 6 \\ &\vdots \\ a_n &\geq 2n \end{aligned}$$

So, $a_1 + a_2 + ... + a_n \geq n(n+1)$.
Thus we have

$$k - n(n+1) = (a_1 - 2) + (a_2 - 4) + ... + (a_n - 2n) + b_1 + b_2 + ..., \tag{3.4}$$

where no b_i is less then (2n+3) and even parts are n in number. (3.4) can be represented by

$$\frac{(-q^{2n+3}; q^2)_\infty q^{n^2+n}}{(q^2; q^2)_n}.$$

So,

$$\sum_{n=0}^{\infty} \frac{(-q^{2n+3}; q^2)_\infty q^{n^2+n}}{(q^2; q^2)_n},$$

gives the total number of partitions of type (3.3). This completes the proof of the interpretation of (3.1).

Interpretation of (3.2)

The number of partitions of a positive integer k into n even parts and distinct odd parts in which no one is less then $(2n+1)$ equals the number of partitions of k into parts $\equiv 1$ or 4 (mod 5).

Proof. Proof of this interpretation is similar to the proof of interpretation of (3.1).

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